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Abstract
The dynamical system approach has recently acquired great importance in the
investigation on higher order theories of gravity. In this talk I review some of
the main results obtained with this method and I consider briefly a number of
further developments.

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1. Introduction

Since the very first proposal of general relativity (GR), modifications of Einstein's theory
of gravity have been introduced for various purposes: from unification of the fundamental
interaction to the explanation of the dark energy phenomenon. In spite of their different nature,
these models have a common drawback: they present even more technical difficulties than
general relativity. This means that it is very hard to obtain an understanding of the physical
properties of these models and, above all, to test them against observations. Consequently,
much effort has been invested into devising ways to overcome these technical issues. A
technique that has proven to be very successful is the so-called dynamical system approach
(DSA) to cosmology. This method was first developed for general relativity and it has led to
beautiful insights into GR cosmologies [1].

In this paper we review some of the recent results obtained from the application of DSA
to fourth order gravity and, in particular, to R^n-gravity [2, 3]. We also discuss very briefly the
perspective of some further applications of this method.

Unless otherwise specified, we will use natural units ($\hbar = c = k_B = 8\pi G = 1$)
throughout the paper and Greek indices run from 0 to 3. The semicolon represents the usual
covariant derivative.

2. R^n-gravity

The action for the gravitational interaction in this theory reads

$$\mathcal{A} = \int \mathrm{d}^4 x \sqrt{-g}[\chi(n) R^n + L_M], \tag{1}$$

where $\chi(n)$ is a positive function of n that reduces to 1 for $n = 1$. For $R \neq 0$, the field equations for this theory can be written as

$$G_{\mu\nu} = T^M_{\mu\nu} + T^R_{\mu\nu} = \frac{\tilde{T}^M_{\mu\nu}}{n\chi(n)R^{n-1}} + \frac{1}{2n}g_{\mu\nu}(1-n)R$$

$$+ \left[(n-1)\frac{R^{;\alpha\beta}}{R} + (n-1)(n-2)\frac{R^{;\alpha}R^{;\beta}}{R^2} \right](g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}), \tag{2}$$

where $\tilde{T}^M_{\mu\nu}$ is the stress energy tensor for standard matter. In this way, the non-Einsteinian part of the gravitational interaction can be modelled as an 'effective fluid' which, in general, has thermodynamic properties different from standard matter[3]. In the following we will apply the DSA to the above equations, first for the Friedmann–Lemaître–Robertson–Walker (FLRW) model and then the Bianchi I model with local rotational symmetry (LRS).

3. Dynamical analysis of the FLRW case

In the FLRW metric and considering standard matter as a perfect fluid, the system (2) takes the form

$$2n\frac{\ddot{a}}{a} + n(n-1)H\frac{\dot{R}}{R} + n(n-1)\frac{\ddot{R}}{R} + n(n-1)(n-2)\frac{\dot{R}^2}{R^2}$$

$$- (1-n)\frac{R}{3} + \frac{\mu}{3n\chi(n)R^{n-1}}(1+3w) = 0, \tag{3}$$

$$H^2 + \frac{\kappa}{a^2} + H\frac{\dot{R}}{R}(n-1) - \frac{R}{6n}(1-n) - \frac{\mu}{3n\chi(n)R^{n-1}} = 0. \tag{4}$$

$$\dot{\mu} + 3H\mu(1+w) = 0, \tag{5}$$

with

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right), \tag{6}$$

where the 'dot' is the derivative with respect to the cosmic time, $H = \dot{a}/a$, κ is the spatial curvature index, w is the barotropic index and we have considered a and R as two independent fields [2].

The form of the above equations suggests the following choice of expansion normalized variables:

$$x = \frac{\dot{R}}{RH}(n-1), \qquad y = \frac{R}{6nH^2}(1-n), \qquad z = \frac{\mu}{3n\chi(n)H^2R^{n-1}}, \qquad K = \frac{\kappa}{a^2H^2}. \tag{7}$$

Differentiating (7) with respect to the logarithmic time $\mathcal{N} = \ln a$, we obtain the autonomous system

$$x' = \frac{2(n-2)y}{n-1} - 2x - 2x^2 - \frac{xy}{n-1} + (1+x)z - 3zw,$$

$$y' = \frac{y}{n-1}[(3-2n)x - 2y + 2(n-1)z + 2(n-1)], \tag{8}$$

$$z' = z\left(2z - 1 - 3x - \frac{2y}{n-1} - 3w \right), \qquad 1 + x - y - z + K = 0.$$

[3] Note that the nature of the matter term and the fact that the sign of R is not necessarily fixed makes this theory fully meaningful only if we consider values of n belonging to the set of the relative numbers $\mathcal{Z}$ and the subset of the rational numbers $\mathcal{Q}$, which can be expressed as fractions with an odd denominator. In the following we will suppose n to belong only to these specific sets.

Table 1. Coordinates of the finite and asymptotic fixed points and solutions for FLRW R^n-gravity in vacuum.

Point	Coordinates	Solution/behaviour
$\mathcal{A}$	$[x, y] = [0, 0]$	$a = a_0 t$
$\mathcal{B}$	$[x, y] = [-1, 0]$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)
$\mathcal{C}$	$[x, y] = \left[\frac{2(n-2)}{2n-1}, \frac{4n-5}{2n-1}\right]$	$a = a_0 t^{\frac{(1-n)(2n-1)}{n-2}}$
$\mathcal{D}$	$[x, y] = [2(1 - n), 2(n - 1)^2]$	$a = a_0 t$
$(\mathcal{A}_\infty, \mathcal{C}_\infty, \mathcal{D}_\infty, \mathcal{F}_\infty)$	$\theta = (0, \pi/2, \pi, 3\pi/2)$	$\|\mathcal{N} - \mathcal{N}_\infty\| = \left[C_1 \pm \frac{C_0}{2}(t - t_0)\right]^2$
$(\mathcal{B}_\infty, \mathcal{E}_\infty)$	$\theta = (\pi/4, 5\pi/4)$	$\|\mathcal{N} - \mathcal{N}_\infty\| = \left[C_1 \pm C_0 \left\|\frac{n-1}{2n-1}\right\|(t - t_0)\right]^{\frac{2n-1}{n-1}}$

Note that the two planes $y = 0$ and $z = 0$ correspond to two invariant submanifolds. This implies no finite global attractor exists for this system. The behaviour of the scale factor corresponding the fixed points can be found using the equation

$$\dot{H} = \left(x_C + \frac{y_C}{n - 1} - z_C - 1\right) H^2. \tag{9}$$

If $n \neq 1$ and $(n - 1)(x_C - z_C - 1) - y_C \neq 0$ this equation can be integrated to give

$$a = a_0(t - t_0)^\alpha \qquad \text{with} \qquad \alpha = \left(1 - x_C - \frac{y_C}{n - 1} + z_C\right)^{-1}. \tag{10}$$

The energy density μ can be written in terms of these variables as

$$\mu = z y^{n-1} H^{2n}, \tag{11}$$

thus for $n > 1$ both the (x, y) and (x, z) planes are invariant vacuum manifolds, but if $n < 1$ the vacuum submanifold is not necessarily compact.

A detailed analysis of this dynamical system is shown elsewhere [2]. Here we will focus on the interval $1.367 \lesssim n < 2$. This interval is suggested by the data coming from WMAP and observations of type Ia supernovae [4].

3.1. The vacuum case

In a vacuum spacetime (i.e. $\mu = 0$), the variable z is identically zero and the third equation of (8) becomes an identity. Let us analyse this case first. Setting $x' = 0$ and $y' = 0$, we obtain the four fixed points shown in table 1. In our interval for n, the solution associate with the point $\mathcal{C}$ represents power law inflation and the only finite attractor. This is an interesting result because this model contains cosmic histories that naturally approach to a phase of accelerated expansion. However, the presence of the invariant submanifolds already mentioned makes this attractor not global and we have to check if in the asymptotic regime other attractors are present.

The idea that the system above might have a nontrivial asymptotic structure comes from the fact that the phase plane is not compact. The asymptotic analysis can be easily performed using the Poincaré approach [5]. We obtained six fixed points which are summarized with their behaviour in table 1. The stability analysis shows that in the interval of n chosen above there is another attractor (this time global): the point $\mathcal{D}_\infty$. Using the asymptotic limit of equation (9) (see [2]), we found that $\mathcal{D}_\infty$ corresponds to a Lemaître type evolution in which the universe reaches a maximum size and then re-collapses. The presence of $\mathcal{D}_\infty$ reduces the measure of the set of initial conditions for which an orbit will approach to $\mathcal{C}$, or in simpler

words it makes this type of cosmic history 'less probable'. A pictorial representation of the whole (compactified) phase space is given in figure 1.

3.2. The matter case

When matter is present we consider the full system (8). Setting $x' = 0$, $y' = 0$, $z' = 0$ and solving for x, y, z we obtain seven fixed points and two fixed subspaces. The first four points lie on the plane $z = 0$ and correspond to the vacuum fixed points. Of the other three fixed points, $\mathcal{G}$ is definitely the most interesting because of its associated solution: $a = a_0 t^{\frac{2n}{3(1+w)}}$, $\mu = \mu_0 t^{-2n}$. The presence of such a fixed point suggests the idea that there could be cosmic histories in this model in which an (unstable) Friedmann-like phase is followed naturally by a phase of accelerated expansion. Applying the standard tools of the dynamical system, we discover that this is actually the case in our interval for n. In fact, for $1.367 \lesssim n < 2$ the point $\mathcal{C}$ is an attractor and the point $\mathcal{G}$ is a saddle; they are both physical and placed in a connected sector of the phase space. Thus in principle a cosmic history like that pictured above is possible. This is also confirmed by numerical investigation of (3)–(5). The remaining thing to check is the presence of other attractors in the asymptotic regimes. The compactification is achieved in the same way as the vacuum case, but the equations obtained are much more complicated. Here we will limit ourselves to say that there are other attractors in the phase space and that their presence reduces the measure of the set of initial conditions that lead to a cosmic history connecting $\mathcal{G}$ and $\mathcal{C}$.

4. Dynamical analysis of the LRS Bianchi I cosmologies

In order to obtain the simplest possible form for the field equations in the LRS Bianchi I metric, we use the $(1 + 3)$ covariant approach to cosmology [6]. In this formalism the cosmological equations can be written as

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 - \frac{1}{2n}R - (n-1)\frac{\dot{R}}{R}\Theta + \frac{\mu}{\chi n R^{n-1}} = 0, \tag{12}$$

$$\frac{1}{3}\Theta^2 - \sigma^2 + (n-1)\frac{\dot{R}}{R}\Theta - \frac{(n-1)}{2n}R - \frac{\mu}{\chi n R^{n-1}} = 0, \tag{13}$$

$$\dot{\sigma} = -\left(\Theta + (n-1)\frac{\dot{R}}{R}\right)\sigma, \tag{14}$$

together with (5). Here, the dot represents the derivative along the observer's velocity field [6], Θ is the volume expansion $\Theta = 3\frac{\dot{a}}{a} = 3H$ and σ is the square root of the magnitude of the symmetric shear tensor σ_{ab}. The set of expansion normalized variables is

$$x = \frac{3\dot{R}}{R\Theta}(n-1), \qquad y = \frac{3R}{2n\Theta^2}(n-1), \qquad z = \frac{3\mu}{\chi n R^{n-1}\Theta^2}, \qquad \Sigma = \frac{3\sigma^2}{\Theta^2}, \tag{15}$$

and the dynamical system can be written as

$$\begin{aligned}
\Sigma' &= -2\left[\left(\frac{2n-1}{n-1}\right)y + z\right]\Sigma, \\
y' &= \frac{y}{n-1}[(2n-1)\Sigma - (2n-1)y + z + (4n-5)], \\
z' &= z\left[(2-3w) - z + \Sigma - \left(\frac{3n-1}{n-1}\right)y\right], \\
1 - \Sigma + x - y - z &= 0.
\end{aligned} \tag{16}$$

Table 2. The finite and asymptotic fixed points and eigenvalues for R^n-gravity in a LRS Bianchi I model. Σ_* is the generic coordinate on the fixed line, Σ_c is an integration constant.

Fixed subspaces	Coordinates	Scale factor	Shear
$\tilde{\mathcal{A}}$	$(\Sigma, y) = \left(0, \frac{4n-5}{2n-1}\right)$	$a = a_0(t-t_0)^{\frac{(1-n)(2n-1)}{(n-2)}}$	$\sigma = 0$
$\mathcal{L}_1$	$(\Sigma, y) = (\Sigma_*, 0)$	$a = a_0(t-t_0)^{\frac{1}{2+\Sigma_*}}$,	$\sigma = \sigma_0 a^{-(2+\Sigma_*)}$
$\mathcal{A}_\infty$	$\phi = 0$	$\tau - \tau_\infty = \frac{1}{\Sigma_c} \ln|t-t_0| + C$	$\sigma = \sigma_0 a^{-(2+\Sigma_c)}$
$(\mathcal{B}_\infty, \mathcal{C}_\infty)$	$\phi = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$|\tau - \tau_\infty| = \left[C_1 \pm C_0\left|\frac{n-1}{2n-1}\right|(t-t_0)\right]^{\frac{2n-1}{n-1}}$	$\sigma = 0$
$\mathcal{D}_\infty$	$\phi = \frac{7\pi}{4}$	$|\tau - \tau_\infty| = [C_1 \pm C_2(t-t_0)]^2$	$\sigma = \sigma_0$

The solution associated with each fixed point can be found via the equations

$$\dot{\Theta} = \left(\frac{n}{n-1}y_i - \Sigma_i - 2\right)\frac{\Theta^2}{3}, \qquad \dot{\sigma} = -\frac{1}{3}(2 + \Sigma_i + y_i + z_i)\Theta\sigma. \tag{17}$$

For $n \neq 1, \sigma \neq 0$ and a RHS different from zero, these equations admit the solution

$$a = a_0(t-t_0)^\alpha, \qquad \alpha = \left(2 + \Sigma_i - \frac{n}{n-1}y_i\right)^{-1}, \tag{18}$$

$$\sigma = \sigma_0 a^\beta, \qquad \beta = -(2 + \Sigma_i + y_i + z_i). \tag{19}$$

As in the previous section we will focus on a specific intervals of n referring the reader to [3] for a general analysis.

4.1. The vacuum case

In order to treat the vacuum case, we set $z = 0$ and we neglect the equation for z as in the FLRW case. Setting $\Sigma' = 0$ and $y' = 0$ we obtain the fixed subspaces and their solutions (see table 2). Note that the fixed point $\tilde{\mathcal{A}}$ represents the same solution associated with the point $\mathcal{A}$. Direct verification with the cosmological equations reveals that this point represents an isotropic ($\sigma_{ab} = 0$) solution. The presence of such a point is particularly interesting because it means that there could be cosmic histories during which the universe goes through an effectively isotropic state. Two interesting cases arise. When $1 < n < 5/4$, the point $\tilde{\mathcal{A}}$ is a repeller and part of the orbits emerging from it approaches the fixed line. This means that in these cosmic histories the universe goes through an effectively isotropic state (see figure 1). Since, as well known there is no isotropic past attractor in anisotropic GR cosmologies, this result implies that in R^n-cosmology, like in brane cosmology, there is no need for special initial conditions for inflation to begin [3]. When $n > 5/4$, $\tilde{\mathcal{A}}$ is an accelerator expansion attractor and all the $y > 0$ orbits start on $\mathcal{L}_1$ and converge to it (see figure 1). So this scenario presents the same smooth transition between a decelerated and accelerated expansion of the FLRW case.

If we analyse the asymptotic regime for this last case we can see that for $n > 5/4$ the point $\mathcal{D}_\infty$ is an attractor (see table 2), but it is in a separate section of the phase space with respect to $\tilde{\mathcal{A}}$, so that $\tilde{\mathcal{A}}$ can be considered a global attractor for the $y > 0$ orbits.

4.2. The matter case

Setting $\Sigma' = 0$, $y' = 0$ and $z' = 0$ in (16) we obtain, together with the vacuum fixed points, other two fixed points, one of which ($\tilde{\mathcal{G}}$) is an isotropic point associated with the same solution
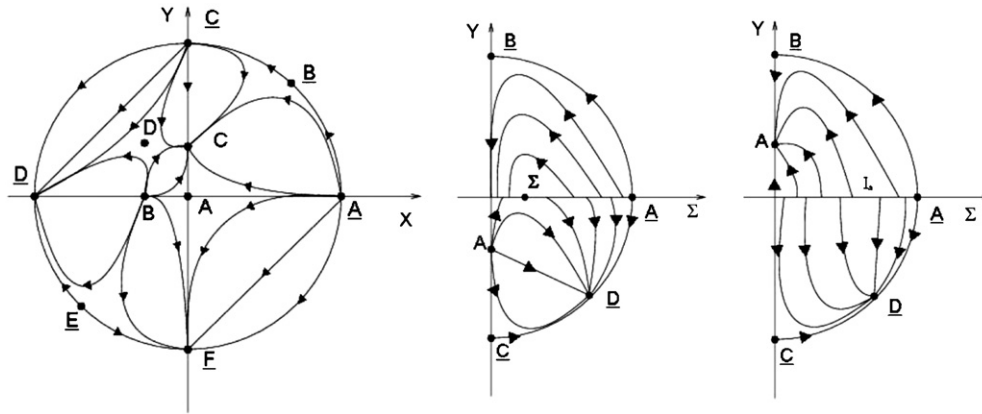
Figure 1. Global (compactified) phase spaces for R^n-gravity. The first picture represents the FLRW case with $1.36 \lesssim n < 2$; the second the Bianchi I case for $1 < n < 5/4$ and the third the Bianchi I $n > 5/4$ case. The standard capital letters represent the finite fixed points and the underlined capital letters represent the asymptotic fixed points. The point Σ is not a fixed point but it is defined as the last attractive point of the fixed line. The curly capital 'l' represents the line $\mathcal{L}_1$.

of $\mathcal{G}$. Let us check if it is possible to have a cosmic history similar to the one we found in section 3. For $n > 5/4$, the fixed line is repulsive nearby the origin, the point $\tilde{\mathcal{G}}$ is unstable and the point $\tilde{\mathcal{A}}$ in an attractor. Since these points are not separated by any invariant subspace, there is in principle an orbit that connects them. Along this orbit, the universe starts in an anisotropic state, isotropize towards a Friedmann-like state and smoothly approach a phase of accelerated expansion. Of course we still have to determine how many other attractors are present (and so how 'probable' this evolution is) so that a detailed asymptotic analysis is required. Using the results of [3], it is possible to show that there are other attractors in the phase space that might influence the global evolution. However, we can conclude that there still is a set of initial conditions with nonzero measure for this type of cosmic history and it surely deserves more study.

5. Conclusion

In this paper we have very briefly reviewed some results obtained applying the DSA to cosmological models based on fourth order gravity. This method has given new insights on these models and it has shown a deep connection between these theories and the cosmic acceleration phenomenon, which is worth for further study.

The prospect for future applications of DSA to higher order gravity can be basically divided into two thrusts. The first one consists in the analysis of more complicated Lagrangians. Some work in this direction has already started in [7] with quadratic gravity. The second one is the generalization to more complicated metrics, which will allow us to consider different physical framework. In the future, both of these thrusts will make it possible not only to develop experimental tests for alternative gravity but also to allow a better understanding of the reasons underlying the success of general relativity.

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